On first principles of viscoelasticity

Igor B. Morozov

Department of Geological Sciences, University of Saskatchewan, Saskatoon, SK S7N 5E2, Canada; tel. 1-306-966-2761, Fax 1-306-966-8593

igor.morozov@usask.ca

Abstract

Fundamental principles of the viscoelastic theory are compared to those of theoretical mechanics, classical theory of elasticity, and also to Biot's theory of saturated porous rock. This comparison shows that viscoelasticity lacks the physical foundation of the variational dynamic principle, which is standard in mechanics. In addition, viscoelasticity does not address the actual physical mechanisms of energy dissipation but uses mathematical generalizations that allow bypassing the detailed physics and encoding the observed attenuation coefficients of *P*- and *S* waves into the elastic moduli of the medium. Therefore, viscoelasticity can be viewed not as a physical theory but rather as a mathematical model of general linear processes. This model is accurate for several known end-member cases, such as the *P*- and *S*-wave propagation in uniform media. However, its predictions for heterogeneous media are based on strong mathematical assumptions and still need to be verified from full physical considerations. At the same time, it appears clear that such assumptions should fail in the general case. Viscoelastic medium properties, such as the generalized bulk and shear moduli, are phenomenological by nature. In each specific case, these properties should be related to combinations of various physical factors. Unfortunately, for deep-Earth conditions, such factors still remain to be identified, and true physical theories of wave attenuation still need to be developed. The development of such theories should affect the existing interpretation of seismic attenuation, as well as its forward and inverse models.

Key words:

Anelasticity; seismic attenuation; mechanics; theory.

Introduction

Most of today's observations of seismic attenuation are explained by the theory of viscoelasticity. This theory is presented in all standard texts on theoretical seismology (e.g., Aki and Richards, 2002; Dahlen and Tromp, 1998), and seismologists tend to believe that it provides a correct, complete, and even the only explanation of seismic attenuation within the Earth. Viscoelastic models were successfully constructed for a vast range of frequencies from those of the Chandler wobble to ultrasonic lab measurements. Nevertheless, in order to understand its significance, seismologists still need to better understand the key principles of this theory.

Viscoelasticity is a vast theoretical subject in itself, and it is closely linked to the concept of the quality factor, Q, its measurement, physical significance, frequency dependence, and relation to geological properties. Some debate about these properties started recently (Morozov, 2008, 2009a, b; Xie and Fehler, 2009; Morozov, 2010a), and it is going to continue in a special forum in October issue of *Pure and Applied Geophysics*. The purpose of the present paper is to offer a part of this discussion to *JOSE* readers and to clarify only one, but fundamental point about the viscoelastic theory, which is critical for understanding many of its applications.

Problem of physical basis for viscoelasticity

The question we consider in this paper is: does viscoelasticity represent a fundamental physical theory or only a mathematical summary of some observed attenuation processes? The answer to this question is important for understanding whether the viscoelastic parameters, such as the Q and the time-retarded moduli can be regarded as fundamental properties of the Earth's material or only as some *ad hoc* parameters that allow modeling the observed seismic-amplitude decays. Unfortunately, this question cannot be answered by considering the data fits, because richly-parameterized mathematical models often fit the data well. Overall, viscoelastic predictions are typically plausible phenomeonologically, but they can rarely be verified by other physical theories.

Therefore, we have to look into the first principles of viscoelasticity and decide whether they are close enough to the level of a true physical theory.

The early models of seismic-wave attenuation in solids, for example, by Knopoff and MacDonald (1960), were completely "physical" in character and focused on finding mechanical, thermo-elastic, ferro-magnetic, or other mechanisms that could explain seismic *Q* observations. These efforts seemed to be somewhat thwarted by the difficulty of finding a physical system that could provide a frequency-independent *Q*. Most mechanisms arising from known dissipative systems led to frequency-dependent quality factors. This was an interesting observation in itself, because the general reason for the pervasive frequency-dependence of modeled *Q* could be the way this factor was defined in seismology (Morozov, 2009c). Notably, the force of "solid friction" proposed by Knopoff and MacDonald (1960) to explain the constant *Q* was already "viscoelastic" in character — it was proportional to the acceleration, and therefore could be viewed as a modification of either the density or the elastic constants.

With growing acceptance of the frequency-dependent *Q* and increasing use of the viscoelastic theory in seismology, the focus changed to finding the general laws of wave-attenuation processes without looking into their detailed mechanics. A mathematical form for such laws was found in Boltzmann's (1874) concept of material memory (*Nachwirkung*; after-effect) combined with the interpretation of attenuation as "imperfect elasticity" (Anderson and Archambeau, 1964). In this model, the stress, σ, at any point in the medium is related to the strain rate, $\dot{\varepsilon}$, taken at the same point in space but at all preceding times (Dahlen and Tromp, 1998):

$$\sigma(t) = \int_{-\infty}^{t} M(t-\tau)\dot{\varepsilon}(\tau)\,d\tau \ , \tag{1}$$

where $M(t)$ are the generalized time-dependent viscoelastic modulus. Function $M(t)$ represents the response of the stress to a step-function strain $\varepsilon = \varepsilon_0\theta(t)$, which is equivalent to a delta-function perturbation of the strain rate. Similarly, the strain at a given point also records the stress-rate history at that point:

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\dot{\sigma}(\tau)\,d\tau \ , \tag{2}$$

where $J(t)$ is the corresponding compliance function. The existence of such convolutional responses is usually inferred from creep experiments, in which a step-function stress, $\sigma = \sigma_0\theta(t)$, is applied, and eq. (2) predicts the measured strain,

$$\varepsilon(t) = \frac{\sigma_0}{M_U}\left[1+\phi(t)\right]. \tag{3}$$

In this expression, $\phi(t)$ is called the creep function, and M_U is the "unrelaxed" (initial-state at $t = 0$) value of modulus M.

The symmetry between eqs. (1) and (2) suggests the first problem with physical significance of viscoelasticity, because it is unclear from them whether it is the strain that causes the stress or vice versa. Such symmetry is typical for this theory, which treats all field variables as linearly related to each other by convolutional laws in the time domain. Generally, it arises from using linear relations between the frequency-domain representations of the strain and stress: $\sigma(\omega) = M(\omega)\dot{\varepsilon}(\omega)$, and $\varepsilon(\omega) = J(\omega)\dot{\sigma}(\omega)$. The symmetry between them is emphasized in Zener's (1948) constitutive law known as the standard linear solid,

$$\sigma + \tau_\sigma\dot{\sigma} = M_R\left(\varepsilon + \tau_\varepsilon\dot{\varepsilon}\right), \tag{4}$$

where M_R is the relaxed (static) modulus, which corresponds to the case of $\dot{\sigma} = 0$ and $\dot{\varepsilon} = 0$. Parameters τ_ε and τ_σ are the relaxation time constants describing the delay in $\varepsilon(t)$ responding to a step-function stress $\sigma = \sigma_0\theta(t)$, and vice versa; these parameters must satisfy $\tau_\varepsilon > \tau_\sigma$ in order to ensure $Q > 0$. This relation is satisfied by many mechanisms that were proposed to explain seismic attenuation.

Along with the above symmetry, there are several more reasons to doubt the physical character of viscoelasticity: i) the use of only two parameters to describe the great diversity of attenuation mechanisms, ii) the lack of an unambiguous definition of energy and the corresponding absence of the Lagrangian or Hamiltonian formulations, and iii) excessive universality of the constitutive equations (1) and (2). In addition, viscoelastic boundary conditions and the correspondence principle lead to incorrect phases of the acoustic impedance in the presence of attenuation contrasts (Morozov, 2010b) and to a violation of the energy balance in the predicted mantle Love-wave attenuation (Morozov, unpublished). However, in the following, let us focus only on the three general reasons i) – iii) above.

Parameterization of attenuation properties

It is the parameterization of the problem that distinguishes the true physical descriptions of various processes from their phenomenological models. To build a physical description, one needs to select the parameters which uniquely describe the "state" of the system, such as the displacements u_i and their time derivatives $\dot{u}_i$ in the Lagrangian model of the following section. Further, physical interactions should be described in terms of these parameters; these interactions also involve parameterization of the physical parameters of the medium, such as its mass density, elastic moduli, and properties controlling the attenuation.

The theory of viscoelasticity takes a very simple view on parameterizing the energy dissipation and assumes that it is caused by modifications to the bulk and shear elastic moduli, both of which acquire the form of $M(t)$ in eq. (1) and lead the corresponding quality factors $Q(\omega) = -\mathrm{Re}M(\omega)/\mathrm{Im}M(\omega)$ (e.g., Aki and Richards, 2002, p. 171). Mathematically, as illustrated below, this approach provides an extremely powerful view on Q as a property of the medium representing the argument of its complex-valued modulus. However, physically, this picture is most worrisome, considering that many factors control the energy dissipation within the Earth, such as fracturing, fluids, saturation, viscosity, porosity, permeability, tortuosity, properties of "dry" friction on grain boundaries and faults, anisotropy, and distributions of scatterers (Bourbié et al., 1987). In resonant-system experiments, both in the lab and using the free oscillations of the Earth, the shapes and structures of the specimen (or of the Earth) also represent major factors not always well accounted for. Most of these factors are only remotely (if at all) related to the seismic velocities or elastic parameters. In addition, as Morozov (2009c) argued, Q cannot be measured as a consistent local property of the medium.

Apparently the only existing treatment of the wave-attenuation problem based on a rigorous physical formulation was given by Biot (1956), who considered the propagation of seismic waves through a fluid-saturated porous rock. This example, also outlined in the following section, suggests three important conclusions: 1) attenuation should be included in neither the kinetic nor elastic energies, and therefore it should not be related to the elastic moduli, 2) functionally, attenuation is more akin to the kinetic energy and can hardly be viewed as "imperfect elasticity," and 3) its descriptions in realistic media require complex, multi-phase models of the medium containing numerous, specific parameters. Although the cumulative quality factors Q_P and Q_S (for *P*- and *S*-waves) or the corresponding Q_K and Q_μ (for the inferred moduli) can be derived from Biot's model, they only relate to the plane-wave solutions and do not replace the actual parameters of the porous medium. Regardless of the physical mechanism of attenuation, Q_K^{-1} and Q_μ^{-1} always remain the same linear combinations of Q_P^{-1} and Q_S^{-1}, which are modeled or measured for plane waves in a uniform medium.

Energy and dynamic principle

When constructing a theory of energy dissipation, the first quantity that needs to be defined is the energy. The concept of energy plays a unique role in physics. Energy is not merely a quantity that is preserved in a class of processes called "conservative;" instead, it is critical in forming the dynamic principle that governs the motion of the system. Unfortunately, seismologists rarely study theoretical mechanics and the classical theory of elasticity (Landau and Lifshitz, 1976, 1986), and therefore these subjects require a brief introduction here.

In theoretical mechanics, *all equations of motion* of any system, no matter how complex it is, are encoded in the *functional form* of a single function, which is called the Lagrangian. This function usually combines the *kinetic* (E_k) *and potential* (E_p) *energies* of the system as $L = E_k - E_p$ (Fig. 1). For example for the elastic field, these energies are

$$E_k = \int \frac{\rho}{2}\dot{u}_i\dot{u}_i dV , \qquad \text{and} \qquad E_p = \int \left(\frac{\lambda}{2}\varepsilon_{kk}\varepsilon_{ll} + \mu\,\varepsilon_{ij}\varepsilon_{ij} \right) dV \tag{5}$$

where u_i is the displacement vector, ε_{ij} is the strain tensor, λ and μ are the Lamé moduli of the medium, ρ is its density, dot denotes the partial derivative in time, and the integrals are taken over the entire volume of the field. Summations over all repeated subscripts are also assumed. Notably, L should be real-valued, which already destroys the prospect of complex-valued λ, μ, or ρ in this theory. Most importantly, the displacements (called "generalized coordinates"), u_i, and their time derivatives, $\dot{u}_i$, are treated as independent variables in L. Consequently, L describes not only the systems that obey the equations of motion (which are sometimes called "on mass shell") but also those which do not represent viable solutions, such as those for which $\dot{u}_i \neq du_i/dt$. Such "off mass shell" fields are very important in wave mechanics.

The Hamilton dynamic principle consists in requiring the stationarity of the *action*, defined as $S = \int L dt$, in respect to arbitrary variations of field $\mathbf{u}$:

$$\delta S = 0 , \qquad \text{or} \qquad \frac{\delta S[u]}{\delta u_i} = 0 . \tag{6}$$

By evaluating these functional derivatives, all equations of motion are generated. By exploring various symmetries of the potential- and kinetic-energy forms, a complete set of conservation laws (such as the conservation of the total energy or momentum) is derived. Once again, the definition of the kinetic and potential energies is crucial and at the same time completely sufficient for describing any physical system. One can say that the Lagrangian and variational equations (6) are practically required from a proper physical theory.

The kinetic and potential energy functions in mechanics are defined from the fundamental principles of time- and translational invariance, linearity, presence of certain degrees of freedom, and types of interactions. These properties are identified for elementary systems, which are more basic than the spring-dashpot combinations commonly used as mechanical analogs in viscoelasticity (e.g., Bland, 1960; Dahlen and Tromp, 1998). For example, the simplest mechanical system is the free mass m, which only possesses the kinetic energy, $E_k = m\dot{x}^2/2$, where $\dot{x}$ is the velocity. For another such system, a particle is subjected to an external force $f(x)$, and its potential energy is $E_p = -\int_{-\infty}^{x} f(x')\,dx'$ regardless of the actual movement. The compete 1D single-body mechanics is therefore embedded in a combination of these energy functions, $L(x,\dot{x}) = E_k - E_p$. Note that it is therefore important not the way the energy is subdivided into E_k and E_p, but rather how $L(x,\dot{x})$ functionally depends on the generalized coordinates and velocities.

Further, let us consider energy dissipation. In mechanics, it is described by an additional function D, which is *not a part* of the Lagrangian and has to be introduced separately, and specifically for each mechanism of dissipation (Fig. 1). Physically, D can be interpreted as the rate of energy dissipation. For example, viscous friction is described by the following Rayleigh dissipation function:

$$D = \frac{\zeta}{2}\dot{x}^2, \tag{7}$$

where ζ is the viscosity constant. The role of D is in serving as a generator of the force of friction: $f_D = -\partial D/\partial\dot{x} = -\zeta\,\dot{x}$, when evaluating the variations of action (6) (see equation in Fig. 1). "Dry" friction with a velocity-independent force $f_D = -f_0\,\mathrm{sgn}(\dot{x})$ would be given by Coulomb dissipation function (gray in Fig. 1),

$$D = f_0|\dot{x}|. \tag{8}$$

Note that in its dependence on velocities, D is functionally similar to the kinetic, but not to the potential energy. This reflects the fundamental observation that friction occurs by relative movements of parts of the system. It appears that if looking for intuitive analogies, we can say that the kinetic energy is the one that dissipates, but not the potential energy, as it is assumed in viscoelasticity.

For an elastic field with dissipation, the dynamics becomes much more complex. Dissipation only occurs in media with at least two "phases" present, and we need to be very specific about using the actual physical mechanisms of deformation. The author is aware of only one case of such kind solved from first principles, which is the theory of saturated porous rock by Biot (1956) and his followers (e.g., Deresiewicz, 1962; Beskos et al., 1989). A very good summary of it this approach was given by Bourbié et al. (1987). In Biot's theory, for the relatively long wavelengths typically encountered in seismology, the volumetric density of the kinetic energy is approximated as quadratic in velocities,

$$E_k = \frac{\rho_u}{2}\dot{u}_i\dot{u}_i + \frac{\rho_w}{2}\dot{w}_i\dot{w}_i + \rho_{uw}\dot{u}_i\dot{w}_i , \tag{9}$$

where $\dot{\mathbf{w}} = \dot{\mathbf{U}} - \dot{\mathbf{u}}$ is the filtration velocity, equal the velocity of the fluid, $\dot{\mathbf{U}}$, relative to that of the rock matrix, $\dot{\mathbf{u}}$. The kinetic matrix-fluid coupling parameters ρ_u and ρ_w above are determined by the porosity and permeability of the rock. For example, ρ_u in this expression is the average density,

$$\rho_u = \rho = \left(1-\phi\right)\rho_s + \phi\rho_f , \tag{10}$$

where ρ_s and ρ_f are the densities of the matrix and fluid, respectively, and ϕ is the porosity. Note that this expression shows that the inertial (i.e., "density") factor is modified in the presence of energy dissipation (Bourbié et al., 1987). This is again contrary to the viscoelastic model (Anderson and Archambeau, 1964).

Similarly to E_k, Biot's dissipation-function density is quadratic in filtration velocity,

$$D = \frac{\eta}{2\kappa}\dot{w}_i^*\dot{w}_i , \tag{11}$$

where η is the fluid viscosity, and κ is the absolute permeability, which depends on the geometry of the pores among other factors. By proceeding with solving the variational equations (6), all *P*- and *S*-wave velocity dispersion laws and attenuation coefficients are obtained. For example, for both the slow and fast *P* waves, spatial attenuation coefficients can be expressed through the normalized functions $\tilde{\alpha}_P\left(\omega/\omega_{\mathrm{B}}\right)$ given in Bourbié et al. (1987):

$$\alpha\left(\omega\right)=\frac{\omega_{\mathrm{B}}}{V_P}\tilde{\alpha}_P\left(\frac{\omega}{\omega_{\mathrm{B}}}\right). \tag{12}$$

Here, ω_{B} is Biot's reference frequency, which depends on the filtering medium:

$$\omega_{\mathrm{B}}=\frac{\phi}{\rho_f \mathrm{K}}. \tag{13}$$

where $\mathrm{K} = \kappa/\eta$ is the hydraulic permeability, or mobility of the fluid within the rock. In the last expression, κ is the absolute permeability, which depends on the geometry of the pores among other factors. Note that ω_{B}^{-1} can be viewed as a kind of a relaxation-time constant, which may be analogous to τ_ε or τ_σ in viscoelasticity (eq. 4) but arises from the properties of the fluid transport through medium and is virtually unrelated to its elasticity.

The above example from Biot (1956) is only given to illustrate two major points. First, rather than assuming an abstract strain-stress memory function, the analysis of energy dissipation within the medium should focus on the specific physical properties, which are typically related to some internal structure and relative movements within the medium. Importantly, this analysis should address the mechanical properties of the medium itself, and not only those of a harmonic wave in it. The Lagrangian approach provides well-established and powerful means for such an analysis. Second, this analysis is in fact very difficult, because many physical properties of the material need to be understood in order to write out the Lagrangian. For deep crustal and mantle conditions, even the basic ideas about parameterizing the Lagrangian similarly to eq. (9) are unclear at present. The solution should be nowhere close to the ease of the viscoelastic approach describing all types of attenuation in all Earth media by extending the only two elastic moduli.

Principle of viscoelasticity

Compared to the above, the viscoelastic theory is constructed differently. In this theory, the energy density function cannot be uniquely defined (see Section 2.1 in Carcione, 2007). Viscoelasticity therefore bypasses the variational principle and starts by directly postulating a constitutive law, i.e., the differential equations of motion. The notion of "stored" energy is introduced only after this law is formulated, and typically only for processes harmonic in time. The stored energy is not partitioned into the kinetic and potential parts; instead, it is defined so that its time derivative corresponds to the dissipated power, but functional form — to the elastic energy (see the preceding reference or Section 1.2 in Borcherdt, 2009). In consequence, the cause of energy dissipation is verbally described as "imperfect elasticity" (Anderson and Archambeau, 1964). However, this association occurs because if only harmonic processes are considered, the functional difference between the velocities ($\dot{u}_i = -i\omega\, u_i$) and displacements ($u_i$) disappears, and E_k, E_p, and D can no longer be separated (Fig. 1). As one can see, the viscoelastic energy definition is constructed backwards—by designing a suitable energy function for a known harmonic solution instead of deriving such solutions from the variational principle.

The absence of consistent energy functions is a fundamental flaw of viscoelasticity as a physical theory. Without a variational dynamic principle (6), viscoelasticity is more of a mathematical model of some known decaying-wave processes than a way to discover the true mechanics of attenuation. Phenomenologically, this model works well and allows explaining the behaviors of complex mechanical and electrical systems in engineering and geodesy; however, in geophysics, we need a way for reaching closer to the inner workings of the physical processes within the Earth.

Instead of the dynamic principle, viscoelasticity uses a procedure for projecting the observed Q values into the medium properties, which can be succinctly summarized as follows. Seismic waves are first assumed to attenuate due to only two anelastic factors. According to the partitioning of strain into the dilatational and shear components, these factors are associated with the corresponding bulk and shear moduli and denoted Q_K^{-1} and Q_μ^{-1}, respectively. To infer a relation of these factors to the observable parameters, two cases of harmonic, elastic P- and S- body waves are considered in a uniform space. For such waves, the corresponding phase velocities are

$$V_P = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}} \qquad \text{and} \qquad V_S = \sqrt{\frac{\mu}{\rho}}\,, \tag{14}$$

where K and μ are the bulk and shear moduli, respectively, and ρ is the density. Next, an attenuation is introduced, and the attenuation coefficients for these waves, α_P and α_S, are shown to be proportional to their respective wavenumbers, k_P and k_S. Therefore, the corresponding quality P- and S-wave factors Q_P and Q_S are defined as

$$\alpha_{P,S} = \frac{Q_{P,S}^{-1}}{2} k_{P,S} . \tag{15}$$

These attenuation coefficients are further included in the complex wavenumbers:

$$\tilde{k}_{P,S} = k_{P,S} + i\alpha_{P,S} \approx k_{P,S} e^{\frac{i}{2Q_{P,S}}} , \tag{16}$$

and the complex phase velocities are also defined in order to preserve the forms of the elastic-case dispersion relations:

$$\tilde{V}_{P,S} = \frac{\omega}{\tilde{k}_{P,S}} \approx V_{P,S} e^{\frac{-i}{2Q_{P,S}}} . \tag{17}$$

By using the correspondence principle, $\tilde{V}_{P,S}$ are then attributed to the medium as its complex wave speeds. This is perhaps the most important, and also the trickiest part of the argument, in which a local physical property of the medium, such as $V_S = \sqrt{\mu / \rho}$ at some point, is equated to the complex plane-wave phase velocity $\tilde{V}_S$ derived for an idealized plane-wave case (Fig. 2). Note that, for example, in a surface wave with frequency ω and wavenumber k, wave speeds $V_{P,S}$ do not equal ω/k practically anywhere in the wave, and the reason for applying eq. (17) remains questionable.

Nevertheless, continuing from eq. (17), we see that for weak attenuation, the P- and S-wave quality factors represent complex arguments of $\tilde{V}_P$ and $\tilde{V}_S$, respectively:

$$\arg \tilde{V}_{P,S} = -\arg \tilde{k}_{P,S} = -\frac{1}{2Q_{P,S}} . \tag{18}$$

To explain the complex-valued material $\tilde{V}_P$ and $\tilde{V}_S$, relations (14) are assumed to hold in the anelastic case, and K and μ are considered as complex-valued. Density ρ is still treated as real based on its intuitive connotation with "imperfect gravity" (Anderson and Archambeau, 1964). Therefore, the two body-wave parameters Q_P and Q_S become mapped into the complex-valued moduli of the medium,

$$\mu = \rho \tilde{V}_S^2, \qquad \text{and} \qquad K = \rho \left(\tilde{V}_P^2 - \frac{4}{3}\tilde{V}_S^2 \right). \tag{19}$$

Finally, the desired Q_K, and shear, Q_μ, "quality factors" are defined from the complex arguments of the two new moduli:

$$Q_K^{-1} = -\arg K, \qquad \text{and} \qquad Q_\mu^{-1} = -\arg \mu . \tag{20}$$

It is noteworthy that Q_K^{-1} and Q_μ^{-1} are closely tied to K and μ, and consequently to the phase velocities not only in the plane-wave problem considered here but also in the general case, such as the forward kernels for free oscillations (Dahlen and Zhou, 2006). Thus, for weak attenuation, we can think of Q_K^{-1} and Q_μ^{-1} simply as linear combinations of the plane-wave Q_P^{-1} and Q_S^{-1} (Fig. 2).

Up to this point, the derivation only represents a way to formally express the end-member *P*- and *S*-wave solutions (14) in terms of K and μ while preserving their elastic-case forms. However, by placing these complex moduli in the viscoelastic constitutive equations (1) and (2), one now becomes able to consider problems to which the original solutions did not apply, such as the surface waves or free oscillations. For example, for Rayleigh waves, their attenuation factor can be expressed as a linear combination of Q_K^{-1} and Q_μ^{-1}, or of Q_P^{-1} and Q_S^{-1}:

$$Q_R^{-1} = mQ_P^{-1} + (1 - m)\, Q_S^{-1}, \tag{21}$$

where m is a function of the Poisson's ratio (Macdonald, 1959). However, as one can see, the above derivation is heuristic and represents a mathematical transformation which only guarantees to correctly reproduce the results of the initial, plane *P*- and *S*-wave observations. For a heterogeneous medium, there seems to be no reason why the Q_K^{-1} and Q_μ^{-1} values inferred for the hypothetical plane-wave cases in Fig. 2b should relate to the medium near point A (Fig. 2a). We still need to seek corroboration for the values of Q predicted in such manner, such as in eq. (21), from models containing the actual mechanics of the media, such as the saturated porous half-space models by Deresiewicz (1960). Note that Beskos et al. (1989) showed that within certain frequency ranges, different types of porosities caused large variations of Rayleigh-wave attenuation, whereas its phase velocity varied only a little.

Regarding the absence of a variational principle in viscoelasticity, an additional note is appropriate. In the more rigorous and complete mechanical treatments, such as the models of free oscillations described by Dahlen and Tromp (1998), the Lagrangian formalism and the variational principle (6) are actually utilized. However, they are only applied to the elastic case, and typically for determining the frequencies of normal oscillations, which we denote ω_n here. To introduce attenuation, the elastic-case expressions for these frequencies, as well as the elastic moduli within the Earth, are extrapolated into the corresponding complex planes (Dahlen and Tromp, 1998), similarly to our example above. This extrapolation is done without the use of any physical "principle" for energy dissipation. Instead, it is based on a mathematical assumption of functional analyticity, which, for example, states that the partial derivatives of $\mathrm{Im}\,\omega_n$ can be obtained from those for $\mathrm{Re}\,\omega_n$ by the Cauchy-Riemann relations. This is a very strong constraint, which automatically reduces the whole attenuation problem to the one for eigenfrequencies. Mathematical abstractions of such level cannot be justified but only *postulated*. However, the physical argument for such a postulate appears unclear, and it should almost certainly fail because of the complexity of the real attenuation problem described above.

Universality

The viscoelastic model (1) and (2) offers a remarkable universality in application to standing- and traveling-wave problems of all types and at all scales. Such universality might indicate its grasp on the most fundamental principles of wave mechanics in solids; however, it may also mean that the theory is actually too general and permissive, and has little connection to the specific physics of attenuation. In the following, we argue that the second of these alternatives appears to be the case.

Viscoelasticity only requires the linearity of equations of motion in respect to the field variables and their time derivatives but does not imply any specific mechanical properties of the medium. Such linearity is commonly present, and consequently viscoelasticity nearly always "works." To illustrate this point, let us look at a somewhat absurd re-formulation of the second Newton's law closely following the strain-stress model in eqs. (1) and (2). While going through this example, note once again that such an effect is only possible because of the focus on *mathematically describing the solution* rather than *analyzing the construction of the dynamic equations* themselves.

Consider the Newton's equation of motion for a single particle,

$$\ddot{x}(t) = \frac{f(t)}{m}, \tag{22}$$

where x is its coordinate, m is the mass, and f is the force. This equation is linear in x, its time derivatives, and f. Consequently its solution can be written as a superposition of responses to step-function forces applied at varying times t_0, $f(t) = f_0\theta(t-t_0)$. For a single step function, we know that the particle accelerates in the direction of the force:

$$x(t) = x_0 + V_0(t - t_0) + \frac{f_0}{2m}(t - t_0)^2, \tag{23}$$

where x_0 and V_0 are the coordinate and velocity at $t = t_0$, respectively. However, in terms of a viscoelastic model, the movement of this particle can be shown as "creep"

$$x(t) = \phi(t - t_0), \tag{24}$$

where $\phi(t)$ is the creep function analogous to the one in eq. (3),

$$\phi(t) = \theta(t)\frac{f_0}{2m}t^2. \tag{25}$$

In equation (24), we dropped the homogenous part of the solution $x(t) = x_0 + V_0t$, in order to make x analogous to the elastic deformation ε. The step-function force $f_0\theta(t-t_0)$ corresponds to a delta-function "force rate," $\dot{f}(t) = f_0\delta(t - t_0)$; this gives

$$x(t) = \int_{-\infty}^{t} J(t-\tau)\dot{f}(\tau)\,d\tau, \tag{26}$$

where $J(t)$ can be called the "retarded compliance function," similarly to the one in (2),

$$J(t) = \frac{\theta(t)}{2m} t^2 \tag{27}$$

Conversely, the equation of motion (22) can be read in the opposite direction, as force resulting from acceleration:

$$f(t) = m\ddot{x}(t) . \tag{28}$$

This gives the force as a "retarded response to velocity,"

$$f(t) = \int_{-\infty}^{t} M(t-\tau)\,\dot{x}(\tau)\,d\tau \ , \tag{29}$$

where $M(t)$ is the "retarded modulus," as in eq. (1):

$$M(t) = m\frac{\delta(t)}{t} , \tag{30}$$

because $\dot{\delta}(\tau) = \delta(\tau)/\tau$.

In the frequency domain, eq. (22) yields a harmonic solution

$$x(\omega) = -\frac{f(\omega)}{m\omega^2} , \tag{31}$$

and also $\dot{f}(\omega) = -i\omega\, f(\omega)$. Therefore, the frequency-domain "compliance" equals

$$J(\omega) = \frac{x(\omega)}{\dot{f}(\omega)} = \frac{1}{im\omega^3} . \tag{32}$$

Similarly, the frequency-domain "complex modulus" is

$$M(\omega) = \frac{f(\omega)}{\dot{x}(\omega)} = -im\omega \ . \tag{33}$$

If we now add a viscous force to (22):

$$\ddot{x}(t) = \frac{f(t) - \zeta\,\dot{x}(t)}{m} , \tag{34}$$

the frequency-domain solution (31) modifies to

$$x(\omega) = -\frac{f(\omega)}{m\omega^2 + i\omega\zeta}, \tag{35}$$

and $J(\omega)$ and $M(\omega)$ become

$$J(\omega) = \frac{1}{\omega^2(-\zeta + im\omega)}, \tag{36}$$

$$M(\omega) = \zeta - im\omega . \tag{37}$$

Finally, the frequency-dependent Q^{-1} can be formally defined:

$$Q^{-1}(\omega) = -\frac{\operatorname{Im} M(\omega)}{\operatorname{Re} M(\omega)} = \frac{m\omega}{\zeta} . \tag{38}$$

However, this result shows the “dissipation” as inversely proportional to ζ/m, which is opposite to what we might expect. This contradiction occurs because the system is in fact non-oscillatory. Nevertheless, the described solution has all the attributes of a “viscoelastic” model.

In summary of this section, a complete set of viscoelastic parameters can be derived for a single-body problem in mechanics, which nevertheless has no real relevance to the problem of wave attenuation in elastic media. As this example shows, the essence of the retarded-moduli description consists in exploiting the linearity of the relations between the strain, stress, and all of their time derivatives. This linearity allows defining $M(\omega)$ and $J(\omega)$ as ratios of these quantities. Nevertheless, these ratios only represent attributes of the solutions to eq. (22) written in the form of convolutional integrals (26) and (29). They do not mean that the second Newton’s law includes memory properties or physically relates the force to velocity, as it might seem, for example, from eq. (29). Viscoelasticity is not a true mechanics but only a *model of it*, similarly to the dashpots and springs commonly used for its own illustrations.

Discussion and conclusions

With the convenient interpretation of material memory, it is nevertheless important to not take this concept too literally. Similarly to the stress-strain relation (2), it is equally correct to say that the particle coordinate, $x(t)$ bears the memory of the force rate, $\dot{f}(t)$ in eq. (26), yet this would be a very awkward and misleading way to present mechanics. A literal interpretation of eq. (29) as the force representing a retarded response to the velocity, which is similar to (1), would be even farther from the meaning of the second Newton's law.

It is important to be not influenced by verbal associations suggested by the traditional, intuitive terminology, such as the notions of "memory," "imperfect elasticity," or "quality factor." Looking at the strain-stress relationships (1) and (2) rigorously, one can see that this memory arises simply from the linearity of the strain-stress relations in all known wave-like solutions. Such solutions can be reproduced by several types of mechanical models consisting of springs and dashpots, as discussed in most texts on viscoelasticity (e.g., Bland, 1960; Carcione, 2007), or mathematically — by convolotional integrals (1) and (2). Nevertheless, this does not mean that the strain is actually caused by the stress, or vice versa, or that such systems contain real memory mechanisms. The "memory" arises only by summarizing two general properties of wave solutions, which: 1) are given by causal, attenuating waves in the time domain, and 2) can be linearly superimposed on each other. Therefore, all constitutive equations reduce to multiplications in the frequency domain and to convolutions in the time domain. Practically any model predicting linear attenuation, regardless of being correct or not, physical or not, Lagrangian or based on some ad hoc relations, can be summarized in the form of constitutive equations (1) and (2).

Thus, our conclusion is that viscoelasticity represents a mathematical model of seismic attenuation process rather than a truly physical theory of it. It can be viewed as a *phenomenological generalization of the empirical strain-stress relations* found in creep experiments and some theoretical models rather than an insight in the actual mechanics of the attenuating medium. The mathematical form of this model is very elegant and versatile, allowing extrapolation of several exact analytical solutions to the general case by using relations (1) and (2) which look like a constitutive law. The model is capable of describing almost every case of the Q-factor observed for a variety of standing and traveling waves. The general reason for its ability to predict "reasonable" Q values appears to be in the use of practically the same quantity, Q_K and Q_μ, as the internal parameter of attenuation. Similar properties, such as the power-law frequency dependence, $Q = Q_0(f/f_0)^\eta$, are attributed to both the observed and in situ quality factors, making the construction of successful forward models relatively easy at any frequencies.

According to such character of the viscoelastic theory, its principal parameters, Q_K and Q_μ , should also be understood as phenomenological quantities. Precise meanings of these parameters can be expressed as follows: “For a given point *A* within the medium, if we consider a uniform medium having all properties as at point *A*, and measure the plane-wave attenuation factors in it, Q_P and Q_S, then Q_K and Q_μ are the quantities obtained by transforming these Q_P and Q_S using eqs. (17), (19), and (20)“ (Fig. 2). A similar definition can be constructed for Q_μ derived from torsional experiments in the lab or free oscillations. Clearly, this definition represents quite a remote abstraction, showing that Q_K and Q_μ are actually “not observable” in a non-uniform medium. When derived from field or lab observations by inverting the viscoelastic equations, these parameters might apply, for example, only to some specific wave types or free-oscillation modes. Considering the complex and poorly-understood physics of seismic attenuation within the deep Earth, the physical significance and the very existence of the in situ Q_K and Q_μ still remains wide open for discussion and research. Once these quantities are better understood, or most likely superseded by new, “physical” in-situ factors responsible for elastic-energy dissipation, the existing interpretations of seismic attenuation, as well as its forward and inverse models for the Earth should undoubtedly change.

Acknowledgments

This study has greatly benefited from several discussions with Bob Nowack. Funding from Canada NSERC Discovery Grant RGPIN261610-03 is gratefully acknowledged.

References

Aki K, Richards PG (2002) Quantitative Seismology. Second Edition, University Science Books, Sausalito, CA.

Anderson DL, Archambeau CB (1964) The anelasticity of the Earth, J Geophys Res 69: 2071–2084

Beskos DE, Papadakis CN, Woo HS (1989) Dynamics of saturated rocks. III: Rayleigh waves, J. Engrg Mech 115: 1017–1034

Biot MA (1956) Theory of Propagation of Elastic Waves in a Fluid-Saturated Porous Solid. J Acoust Soc Am 28: (a) 168–178, (b) 179–191

Bland D (1960) The theory of linear viscoelasticity, Pergamon Press.

Boltzmann L (1874) Zur Theorie der elastischen Nachwirkung. Sitzungsber Math Naturwiss Kl Kaiserl Wiss 70: 275

Borcherdt RD (2009) Viscoelastic waves in layered media. Cambridge Univ Press, Cambridge

Bourbié T, Coussy O, Zinsiger B (1987) Acoustics of porous media. Editions TECHNIP, France, ISBN 2710805168

Carcione JM (2007) Wave fields in real media: Wave propagation in anisotropic anelastic, porous, and electromagnetic media. Second Edition, Elsevier, Amsterdam.

Dahlen FA, Tromp J (1998) Theoretical global seismology. Princeton Univ. Press, Princeton, NJ

Dahlen FA, Zhou Y (2006) Surface-wave group-delay and attenuation kernels. Geophys J Int 165: 545–554

Deresiewicz H (1960) The effect of boundaries on wave propagation in a liquid-filled porous solid: I. Reflection of plane waves at a free plane boundary (non-dissipative case). Bull Seism Soc Am 50: 599–607

Knopoff L, MacDonald GJF (1960) Models for acoustic loss in solids. J Geophys Res 65: 2191–2197

Landau LD, Lifshitz EM (1976) Course of theoretical physics, Volume 1: Mechanics. Third edition, Butterworth-Heinemann, ISBN 0 7506 2896 O

Landau LD, Lifshitz EM (1986) Course of theoretical physics, Volume 7: Theory of elasticity. Third edition, Butterworth-Heinemann, ISBN 0 7506 2633 X

Macdonald JR (1959) Rayleigh-wave dissipation functions in low-loss media. Geophys J 2: 132–135.

Morozov IB (2008) Geometrical attenuation, frequency dependence of Q, and the absorption band problem. Geophys J Int 175: 239–252

Morozov IB (2009a) Thirty years of confusion around "scattering Q"? Seism Res Lett 80: 5–7

Morozov IB (2009b) Reply to "Comment on 'Thirty years of confusion around 'scattering Q'?" by J. Xie and M. Fehler. Seism Res Lett 80: 648–649

Morozov IB (2009c) On the use of quality factor in seismology. AGU Fall Meeting, San Francisco, CA, Dec 14–18, 2009, S44A-02.

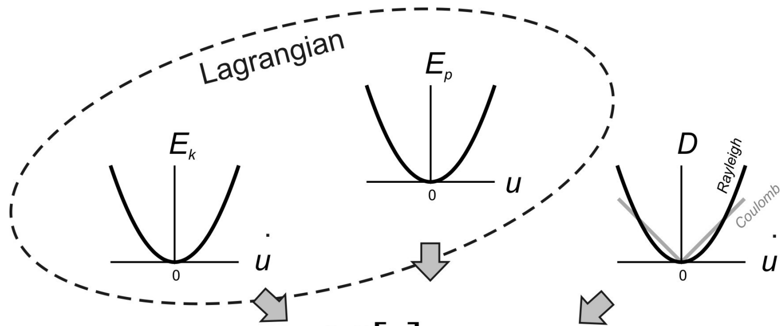
Morozov IB (2010a) On the causes of frequency-dependent apparent seismological *Q*. Pure Appl. Geophys: doi 10.1007/s00024-010-0100-6

Morozov IB (2010b) Anelastic acoustic impedance and the correspondence principle. Geophys Prosp: doi 10.1111/j.1365-2478.2010.00890.x

Xie J, Fehler M (2009) Comment on "Thirty years of confusion around scattering *Q*" by Igor. B. Morozov. Seism Res Lett 80: 646–647

Zener CM (1948) Elasticity and Anelasticity of Metals. University of Chicago Press.

Figures



$$\frac{\delta S[u]}{\delta u}+\frac{\partial D}{\partial \dot{u}}=0$$

Fig. 1. Schematic construction of classical mechanics using functional forms of the kinetic (E_k) and potential (E_p) energies and the dissipation function, D. E_k and E_p are usually quadratic in $\dot{u}$ and u, respectively, but a broad variety of functional forms may be considered for D.

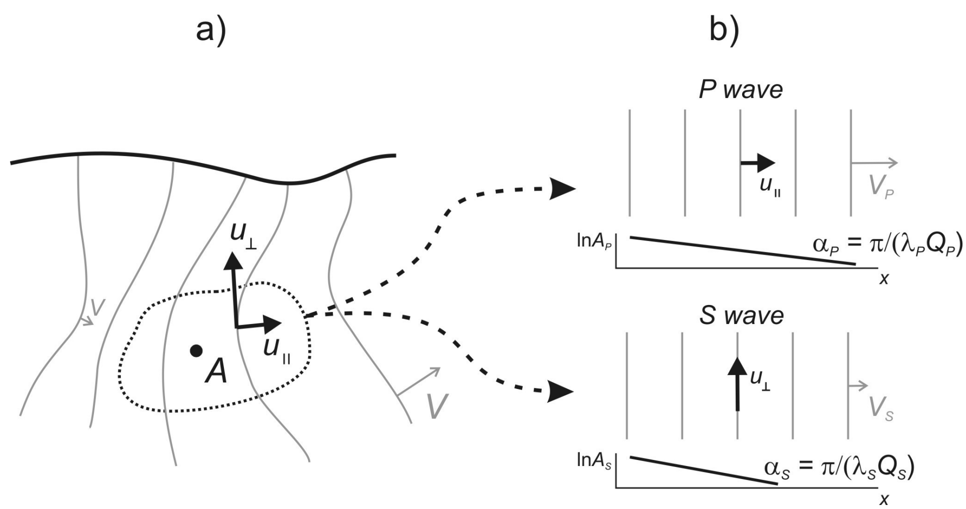


Fig. 2. Rigorous meanings of viscoelastic parameters Q_K and Q_μ: a) in a heterogeneous medium, wave velocities, wavefronts, and amplitudes are variable and depend on numerous parameters in a complex way; b) the vicinity of point A is extrapolated into an "equivalent" uniform-space model, in which the P and S waves are studied. The derived spatial attenuation coefficients, α_P and α_S, are transformed into quality factors, Q_P and Q_S, which are further transformed into Q_K and Q_μ attributed to point A.