

Thermoelastic relaxation in cylindrical material specimens

Here, let us estimate the thermoelastic relaxation in uniform but grainy materials and cylindrical bodies. We will follow Landau and Lifshitz, 1986, §35. If the heat could not transfer during deformation, the adiabatic temperature perturbation due to dilatational deformation Δ would be:

$$T'_0 = -\frac{TK_A\alpha}{\rho C_p} \Delta. \quad (1)$$

However, in the presence of heat exchange, temperature variations are lower and denoted by T' below. Once this quantity is determined, the mechanical-energy dissipation becomes:

$$-\dot{E}_{\text{mech}} = \frac{\kappa}{T} \int [\vec{\nabla} T']^2 dV. \quad (2)$$

Depending of the frequency, the values of T' and of the volume V contributing to dissipation integral (2) vary. Let us consider two practical limits.

1) Low-frequency limit $\omega \ll \mathcal{G}/h^2$

At low frequencies, the entire grain volume participates in heat transfer, and the temperature T' within the grain is approximately uniform. Considering harmonic deformation (for example, $\Delta \propto \cos \omega t$) the average heat production rate by deformation per unit volume equals $\dot{T}'_0 C_v \sim \omega T'_0 C_v$. The heat dissipated by heat conduction is therefore: $\text{div } \mathbf{q} = \kappa \Delta T' \approx \kappa T' / h^2$. By equating these two quantities, we see that temperature variation is proportional to frequency and strain (as $T'_0 \propto \Delta$ in (1)):

$$T' \approx T'_0 \omega \frac{h^2}{g}. \quad (3)$$

This means that temperature variation is proportional to the *strain rate*. From (2), the energy dissipation rate is:

$$-\dot{E}_{\text{mech}} = \frac{\kappa}{T} \left\langle [\bar{\nabla} T']^2 \right\rangle V \approx \frac{\kappa}{T} \left(\frac{T'}{h} \right)^2 V \approx \frac{\kappa}{T} \left(\frac{T'_0 h}{g} \right)^2 \omega^2 V. \quad (4)$$

This dissipation rate is proportional to the squared strain rate and can therefore be described by an equivalent Newtonian viscosity, η_{thermo} :

$$-\dot{E}_{\text{mech}} = \eta_{\text{thermo}} V \Delta^2 \omega^2, \quad (5)$$

where

$$\eta_{\text{thermo}} = \frac{\kappa}{T} \left(\frac{T'_0 h}{g \Delta} \right)^2 = \frac{T}{\kappa} K_A^2 \alpha^2 h^2. \quad (6)$$

For polycrystalline materials (apparently metals), Landau and Lifshitz (1986) mention that this quantity is much larger than the viscosity of the component crystallites. For Plexiglas and polycrystalline olivine, its values are given in Table 1. Note that this viscosity quickly increases with grain size. For olivine at mantle conditions, this viscosity is very large, causing unrealistically high damping ($\tan \phi \gg 1$; Table 1). This shows that the low-frequency limit is likely never reached in seismic observations. By contrast, for “fine-grained” Plexiglas at room temperature, the low-frequency thermoelastic viscosity is low ($\tan \phi \approx 0.01$ at $\sim 1\text{Hz}$; Table 1).

From the expression for thermoelastic viscosity, the corresponding phase lags are:

$$\tan \phi = \frac{\omega \eta_{\text{thermo}}}{K}. \quad (7)$$

2) Intermediate-frequency case $\mathcal{G}/h^2 \ll \omega \ll c/h$

Let us try estimating the parameters of thermoelastic relaxation in cylindrical specimens under harmonic axial compression/tension. Two regimes need to be considered:

- 1) The material is grainy with grain size h , and the dissipation occurs near grain boundaries;
- 2) The material is uniform, and dissipation occurs near the surface of the cylinder, where it is in contact with fluid or gas (argon in Jackson's experiments).

In both of the above cases, for periodic loading/unloading, temperature variations (1) occur within the bulk of material. Near the boundary (of a grain or the whole specimen), these anomalies decay within a thin boundary layer of thickness δ :

$$\delta \approx \sqrt{\frac{\mathcal{G}}{\omega}}. \quad (8)$$

where $\mathcal{G} = \kappa / (\rho C_p)$ is the thermometric conductivity. The temperature gradient within this layer is approximately $\nabla T \approx T'_0 / \delta$. The thermoelastic energy dissipation is then:

$$-\dot{E}_{\text{mech}} = \frac{\kappa}{T} \int [\nabla T]^2 dV \approx \frac{\kappa}{T} \left(\frac{T'_0}{\delta} \right)^2 V \beta. \quad (9)$$

where V is the volume and β is its relative fraction affected by dissipation. For low-

frequency forced oscillations, the total energy is approximately equal the elastic energy:

$$\bar{E}_{\text{mech}} \approx \frac{K}{2} \Delta^2 V, \quad (10)$$

and therefore the damping coefficient is:

$$\chi = \frac{-\dot{\bar{E}}_{\text{mech}}}{2\bar{E}_{\text{mech}}} \approx \beta \frac{\kappa}{TK} \left(\frac{T'}{\Delta\delta} \right)^2 = \frac{\beta}{\delta^2} \frac{\kappa T \alpha^2 K}{(\rho C_p)^2}. \quad (11)$$

Finally, the phase lag equals:

$$\tan \phi = \frac{\chi}{\omega} \approx \frac{\beta}{\omega \delta^2} \frac{\kappa T \alpha^2 K}{(\rho C_p)^2}. \quad (12)$$

For a cylinder of radius R and length L , parameter β equals $\beta \approx 6\delta/h$ in the case

1) above (dissipation on spherical grain boundaries) and $\beta \approx 2\delta/R$ for case 2)

(dissipation on the surface of the whole specimen). Therefore, the phase lag is:

$$\tan \phi \approx \frac{6}{\omega h \delta} \frac{\kappa T \alpha^2 K}{(\rho C_p)^2} \text{ for grainy material,} \quad (13a)$$

$$\tan \phi \approx \frac{2}{\omega R \delta} \frac{\kappa T \alpha^2 K}{(\rho C_p)^2} \text{ for uniform material.} \quad (13b)$$

Although the estimate of δ above is order-of magnitude type and the value of h may be relatively uncertain, we still keep geometrical factors 6 and 2 in these expressions.

In either of these cases, the frequency of the absorption peak, ω_0 , can be estimated from (8) matching the characteristic dimension of the problem:

$$\omega_0 \approx \frac{\mathcal{G}}{h^2} \text{ for grainy material,} \quad (14a)$$

$$\omega_0 \approx \frac{g}{R^2} \text{ for a uniform cylinder.} \quad (14b)$$

For two typical experimental cases, the above quantities are summarized in Table 1. As this Table shows, spectral peaks in Plexiglas can be explained by a finely grained structure. By contrast, it appears that when working with small samples of olivine aggregates at high temperature, the dissipation likely occurs through the surface of the whole specimens.

Estimating grain size

It looks that grain size can be estimated by making the low-frequency and high-frequency expressions (7) and (13) meet at the frequency of absorption peak, ω_0 .

Although none of these equations are valid at the vicinity of point $\omega = \omega_0$, this extrapolation seems to produce reasonable estimates, and it also creates a continuous $\tan\phi(\omega)$ curve. Equating (7) and (13) at frequency ω_0 , we obtain, after some trivial algebra:

$$h = \frac{1}{\sqrt{\omega_0}} \left[\frac{6\kappa^2}{\sqrt{g}(\rho C_p)^2} \right]^{\frac{1}{3}}. \quad (15)$$

These values are shown as “grain sizes” in Table 1.

Table 1. Thermoelastic dissipation models for two typical experiments with cylindrical material specimens.

<i>Material</i>	<i>Plexiglas</i>	<i>Olivine</i>
T , K	300	1200
Dimensions	$R = 7.6$ cm, $L = 25$ cm	$R = 0.5$ cm, $L = 6$ cm
Frequency of spectral peak, f_0 , Hz	2.5	1/1000 (1000-s period, assumed)
K , GPa	2.96	107.8
ρ , kg/m ³	1190	3251
C_p , J/K/kg	1470	1236
κ , W/m/K	0.19	2.0
α , K ⁻¹	$15 \cdot 10^{-5}$	$3.8 \cdot 10^{-5}$
Testing frequency, f , Hz	5.0	1/20 (20-sec period)
<i>General thermoelastic parameters:</i>		
δ , mm	0.06	1.2
$\Delta\kappa$	0.011	0.046
Q_{\min}	137	34
<i>Higher-frequency limit:</i>		
χ for uniform cylinder, Hz	$5.5 \cdot 10^{-4}$	$7.4 \cdot 10^{-3}$
χ for grainy cylinder, Hz	0.84	$6.8 \cdot 10^{-3}$
$\tan\phi$ for uniform cylinder	$1.8 \cdot 10^{-5}$	0.023
$\tan\phi$ for grainy cylinder	0.027	0.022
Grain size, mm	0.15	16
<i>Low-frequency limit:</i>		
η_{thermo} , Pa·s	$7.1 \cdot 10^6$	$2.6 \cdot 10^{12}$
$\tan\phi$ (characteristic values)	0.015 at 1 Hz	1.53 at 100-s period
<i>Comparing the low- and higher-frequency limits:</i>		
$\tan\phi_{\text{thermo}}(\omega_b)$	0.038	7.7